Title	Recovery of Vanishing Cycles by Log Geometry : Case of Several Variables
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Citation	代数幾何学シンポジューム記録 (1996), 1996: 198-205
Issue Date	1996
URL	http://hdl.handle.net/2433/214646
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

代数幾何学シンポジウム記録
1996年度　pp.198-205

Recovery of Vanishing Cycles by Log Geometry: Case of Several Variables

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December, 1996

Abstract

This article is a generalization of the author's work [U] to the case of several variables. We first construct compatible actions of monoid S on a "several-variables-version of semi-stable degeneration of pairs" and on the associated log topological spaces introduced by Kato and Nakayama in [KN]. Here S is the product of the unit interval and the unit circle. Then we show that the associated log topological family is locally topologically trivial over the base, i.e., the associated log topological family recovers the vanishing cycles of the original degeneration. Using this result together with the theory of canonical extensions by Deligne [D], we introduce two types of integral structure of the variation of mixed Hodge structure associated to "several-variables-version of semi-stable degeneration of pairs". We only sketch the proof here. The complete proof will appear soon somewhere.

1 Log Structures

Let $X \subset D$ be a d-dimensional complex manifold and a divisor with normal crossings. The associated *fine saturated log structure* (cf. [K]) is defined by

$$\mathcal{M}_X := \{f \in \mathcal{O}_X \mid f \text{ is invertible outside } D\} \overset{\alpha}{\hookrightarrow} \mathcal{O}_X.$$

Let T be a point $\operatorname{Spec}\mathbf{C}$ with a log structure

$$\mathbf{R}_{\geq 0} \times \mathbf{C}_1 \to \mathbf{C}, \quad (r, u) \mapsto ru,$$

where $\mathbf{C}_1 \subset \mathbf{C}$ is the unit circle. Notice that this log structure is not fine saturated. K. Kato and C. Nakayama introduced in [KN] a *log topological space* $X^{\log}$ as the set of T-valued points in the category of log schemes:

$$X^{\log} := \operatorname{Hom}(T, X) \xrightarrow{\tau_X} X, \quad \text{forgetting morphism.}$$

1991 *Mathematics Subject Classification.* Primary 14C30; Secondary 14D07, 32G 20. Partly supported by the Grants-in-Aid for Scientific Research (1) (B): 08304002, the Ministry of Education, Science and Culture, Japan

Let $\tilde{x} \in X^{\log}$ and $x := \tau_X(\tilde{x})$. Choose a local coordinates $z_1, \ldots, z_d$ at $x \in X$ such that D has a local equation $\prod_{1 \le i \le s(x)} z_i^{m(i)}$, $m(i) \ge 1$. Then we see that

$$\mathcal{M}_{X,x} = \coprod \Big\{ \mathcal{O}_{X,x}^{\times} \prod_{1 \le i \le s(x)} z_i^{b(i)} \;\Big|\; b \in \mathbf{N}^{s(x)} \Big\} \simeq \mathcal{O}_{X,x}^{\times} \oplus \mathbf{N}^{s(x)}, \quad \text{where } \mathbf{N} := \mathbf{Z}_{\ge 0}.$$

$$X^{\log} \overset{\text{locally}}{\simeq} (\mathbf{R}_{\ge 0})^{s(x)} \times (\mathbf{C}_1)^{s(x)} \times \mathbf{C}^{d-s(x)} \xrightarrow{\tau_X} X \overset{\text{locally}}{\simeq} \mathbf{C}^d,$$

$$\tau_X((r_i, u_i)_{1 \le i \le s(x)}, (z_j)_{s(x)+1 \le j \le d}) = ((r_i u_i)_{1 \le i \le s(x)}, (z_j)_{s(x)+1 \le j \le d}),$$

where $r_i := |z_i|$ and $r_i u_i := z_i$. This induces a topology on the set $X^{\log}$, and $\tau_X : X^{\log} \to X$ can be regarded as a real blowing-up (cf. [M]) and $X^{\log}$ as a manifold with corners (cf. [AMRT]).

Example (1.1) Let Δ be the open unit disc in the complex plane, and H the upper half plane. Let $\exp 2\pi\sqrt{-1}(\ ) : H \to \Delta^*$ be the universal cover of the punctured disc. Then the pair $(\Delta, \{0\})$ induces the following diagram:

$$\begin{array}{ccccc}
H & \subset & \hat{H} & := & \mathbf{R} + \sqrt{-1}(\mathbf{R}_{>0} \coprod \{\infty\}) \\
\Big\downarrow & & \downarrow & & \\
 & & \Delta^{\log} & \simeq & \hat{H}/\mathbf{Z} \\
 & & \downarrow & & \\
\Delta^* & \subset & \Delta. & &
\end{array}$$

2 Recovery of vanishing cycles

Let $n \ge 1$ and $a(k)$ $(-1 \le k \le n)$ be integers such that

$$0 = a(-1) \le a(0) < a(1) < \cdots < a(n). \tag{2.1}$$

Set

$$A := \{1, 2, \ldots a(n)\}, \quad A(k) := \{a(k-1)+1, \ldots, a(k)\} \quad (0 \le k \le n). \tag{2.2}$$

Let

$$f : X \to P \tag{2.3}$$

be a proper, flat morphism of a d-dimensional complex manifold X to a polydisc $P := \Delta^n$ with coordinates $t_1, \ldots, t_n$. Let B_k be the divisor on P defined by $t_k = 0$, and set $B := \sum_{1 \le k \le n} B_k$. Set $D := f^*B$ and let

$$f^* B_k =: \sum_{i \in A(k)} m(i) D_i \qquad (1 \le k \le n) \tag{2.4}$$

be the irreducible decomposition. Let $Y = \sum_{i \in A(0)} D_i$ be a divisor on X, flat with respect to f. We assume that

$$Y + D = \sum_{i \in A(0)} D_i + \sum_{1 \le k \le n} \sum_{i \in A(k)} m(i) D_i \tag{2.5}$$

is a divisor with simple normal crossings whose distinct prime divisors are D_i $(i \in A)$. Notice that, locally on the base space, we can reduce any proper, flat family with a flat divisor to the above setting by blowing-ups. The fine saturated log structures associated to the pairs $X \supset D$, $Y \supset D \cap Y$ and $P \supset B$ induce a commutative diagram:

$$\begin{tikzcd}(X \supset Y) \arrow[d, "f"'] & (X^{\log} \supset Y^{\log}) \arrow[l, "\tau_X"'] \arrow[d, "f^{\log}"] \\ P & P^{\log}. \arrow[l, "\tau_X"']\end{tikzcd} \tag{2.6}$$

Let $[0,1] \subset \mathbf{R}$ be the unit interval which is regarded as a monoid by multiplication. The monoid

$$S := ([0,1] \times \mathbf{C}_1)^n \tag{2.7}$$

has natural actions on the polydisc P and on $P^{\log}$. These actions can be lifted to the diagram (2.6), and we have

Theorem 1 *In the above notation, the family of open spaces*

$$\overset{\circ}{f}{}^{\log}\colon (X^{\log} - Y^{\log}) \to P^{\log}$$

is locally topologically trivial over the base $P^{\log}$. This means that $\overset{\circ}{f}{}^{\log}$ recovers the vanishing cycles of the degenerate family

$$\overset{\circ}{f}\colon (X - Y) \to P.$$

We will sketch the construction of the liftings of S-actions to the diagram (2.6) and the proof of Theorem 1 in Section 4 below.

3 Integral structure of degenerate VMHS

We use the notation in Section 2. Here we assume moreover that D is reduced. Then, it can be verified that

$$\mathcal{V} := R^q f_* \Omega^\bullet_{X/P}(\log(Y + D)) \tag{3.1}$$

is the canonical extension of Deligne [D, (5.2)] of $\mathcal{V}|P^*$, $P^* := P - B$, whose Gauss-Manin connection ∇ is obtained as the differential $d_1 : E_1^{0,q} = \mathcal{V} \to E_1^{1,q} = \Omega^1_P(\log B) \otimes_{\mathcal{O}_P} \mathcal{V}$ of the spectral sequence of hypercohomology of the complex $\Omega^\bullet_X(\log(Y + D))$ with respect to a filtration $G^k := f^*\Omega^k_P(\log B) \wedge \Omega^\bullet_X(\log(Y + D))[-k]$.

The locally constant sheaf of $\mathbf{C}$-modules $\mathrm{Ker}(\nabla|P^*)$ lifts to $\tau_P^{-1}(P^*)$ and extends one on $P^{\log}$. We denote the latter by $L'_{\mathbf{C}}$. On the other hand, by Theorem 1, we have locally constant sheaf of $\mathbf{Z}$-modules on $P^{\log}$:

$$L_{\mathbf{Z}} := R^q(\overset{\circ}{f}{}^{\log})_*\mathbf{Z}. \tag{3.2}$$

By construction, $L'_{\mathbf{C}}$ and $\mathbf{C} \otimes_{\mathbf{Z}} L_{\mathbf{Z}}$ coincide on $\tau_P^{-1}(P^*)$, hence they coincide on whole $P^{\log}$ because they are locally constant.

Let $N_i := \log \gamma_i$ $(1 \le i \le n)$ be the monodromy logarithms of $L_{\mathbf{Z}}$ induced by the action of the group $(\mathbf{C}_1)^n$ on $P^{\log}$. Let $\varpi : \hat{H}^n \to P^{\log}$ be the universal covering (cf.

Example (1.1)) and let $l_1, \dots, l_n$ be coordinates on $\hat{H}^n$ with $\exp(2\pi\sqrt{-1}l_i) = t_i$. Choose a flat frame $e_1, \dots, e_r$ of $\varpi^{-1}L_{\mathbf{Z}}$ and modify

$$\tilde{e}_j := \exp\Big(-\sum_{1\le i\le n} l_i N_i\Big)\cdot e_j \quad (1 \le j \le r). \tag{3.3}$$

Then, this drops to a single-valued frame of $\mathcal{O}_P^{\log} \otimes_{\mathbf{Z}} L_{\mathbf{Z}}$ on $P^{\log}$, where $(\mathcal{O}_P^{\log})_{\tilde{t}} := \mathcal{O}_{P,t}[l_1, \dots, l_n]$ for $\tilde{t} \in P^{\log}$ and $t = \tau_P(\tilde{t}) \in P$. Hence this still drops to a frame of $\mathcal{V}$ on P. We also denote this frame of $\mathcal{V}$ by the same symbol $\tilde{e}_1, \dots, \tilde{e}_r$.

It is easy to see, by the definition (3.2), that under the identification

$$\mathbf{C} \otimes_{\mathbf{Z}} (\varpi^{-1}L_{\mathbf{Z}})(h) \xrightarrow{\sim} \mathcal{V}(O), \quad \tilde{e}_j(h) \mapsto \tilde{e}_j(O), \tag{3.4}$$

where $h \in \hat{H}^n$ and $O \in P$ the origin, we have

$$N_i = -2\pi\sqrt{-1}\mathrm{Res}(t_i = 0)(\nabla) \quad \text{(cf. [D, (II.1.17), (II.5.2)])}. \tag{3.5}$$

Thus we have

Theorem 2 *In the notation of Section 2, we assume moreover that D is reduced. Then $\mathcal{V}$ has two types of integral structure:*

(i) $$\mathcal{O}_P^{\log} \otimes_{\mathbf{Z}} L_{\mathbf{Z}} \simeq (\tau_P)^*\mathcal{V} \quad \text{on} \quad P^{\log}.$$

The local monodromies are induced by $(\mathbf{C}_1)^n$-action on $P^{\log}$.

(ii) $$\mathcal{O}_P \otimes_{\mathbf{Z}} (\tau_P)_* R^q(\overset{\circ}{f}{}^{\log})_*(\overset{\circ}{f}{}^{\log})^{-1}\mathbf{Z}[l_1, \dots, l_n] \simeq \mathcal{V} \quad \text{on} \quad P.$$

The monodromy logarithms are given by $-2\pi\sqrt{-1}Res(t_i = 0)(\nabla)$ $(1 \le i \le n)$.

Remark (3.6) (i) $\mathbf{C} \otimes_{\mathbf{Z}} L_{\mathbf{Z}}$ and $(\mathcal{V}, \nabla)$ correspond under the log Riemann-Hilbert correspondence in [KN], by using the monodromy weight filtration in [CK] in case $Y = \emptyset$ and in general case the convolution of the relative monodromy weight filtrations in [SZ] or the weight filtration constructed in [F].
(ii) The author was communicated by Morihiko Saito, on May 24, 1996, that there is a correction of [St, (5.9)] in [Sa, 4.2].
(iii) Fujisawa has obtained some integral structure on $\mathcal{V}$ in different method in [F].

4 Outline of Proof of Theorem 1

The proof is analogous to the argument of Clemens [C], but there are some points in the proof of [C, Theorem 5.7] which are not clear. The readers can find a complete proof in the case of $\dim P = 1$ in [U].

We use the notation in Section 2. For $I \subset A$, we denote

$$D_I := \bigcap_{i\in I} D_i, \qquad I(k) := I \cap A(k) \qquad (0 \le k \le n).$$

The following proposition plays a key role.

Propositon 3 *In the above notation, shrinking the polydisc P, we have the following:*
(a) *There exist a family $\{U_I\}_{I \subset A}$ of open tubular neighborhoods U_I of D_I and a family $\{\pi_I : U_I \to D_I\}_{I \subset A}$ of C^∞ projections which satisfy*

(i) $\quad U_I \cap U_J = U_{I \cup J},$

(ii) $\quad \pi_I \circ \pi_J | U_I = \pi_I \quad$ *for* $\quad I \supset J.$

(b) *There exists a family $\{z_i\}_{i \in A}$ of C^∞ global equations z_i of D_i in X which has the following properties:*

(iii) *If $J \subset A - A(0)$, $x \in D_J$ and $F := \pi^{-1}(x)$, then $\{z_j | F\}_{j \in J}$ forms a system of holomorphic coordinates on F and*

$$\prod_{j \in J(k)} z_j^{m(j)} = (\text{constant})\, t_k \circ f \quad \text{on} \quad F \quad (1 \le k \le n),$$

where the (constant) *depends only on F and on the choice of the z_j and of the t_k.*

(iv) *For $i, j \in A$ with $i \neq j$, z_i is constant on each fiber of $\pi_j : U_j \to D_j$.*

We omit here the proof of this proposition, because it is rather complicated though elementary and also the argument is essentially the same as in the case of $\dim P = 1$ (see [U, §2], in this case). In order to lift the action of monoid $S = ([0,1] \times \mathbf{C}_1)^n$ to the whole diagram (2.6), we should prepare two more things.

For each integer $1 \le k \le n$ and a number $0 \le \delta < 1$, let

$$\begin{aligned} C(k) &:= [0,1]^{a(k)-a(k-1)} \quad \text{unit cube in } \mathbf{R}^{a(k)-a(k-1)}, \\ C(k)_\delta &:= \Big\{ (r_i)_{i \in A(k)} \in C(k) \;\Big|\; \prod_{i \in A(k)} r_i^{m(i)} = \delta \Big\}, \\ E(k)_\delta &:= \bigcup_{\delta' \in [0,\delta]} C(k)_{\delta'}. \end{aligned} \tag{4.1}$$

For each $i \in A - A(0)$, we choose a number

$$0 < \varepsilon_i < 1 \tag{4.2}$$

and a C^∞ function

$$\varphi_i : [0,1] \times [0,1] \to [0,1] \tag{4.3}$$

which have the following properties:

$$\begin{aligned} &\text{If } r \ge \varepsilon_i \text{ then } \varphi_i(s,r) = r. \\ &\text{For all } r,\ \varphi_i(1,r) = r. \\ &(\partial^p \varphi_i / \partial s^p)(0,0) = 0 \quad \text{for all } p \ge 0. \\ &(\partial \varphi_i / \partial s)(s,r) > 0 \quad \text{if } s > 0 \text{ and } 0 < r < \varepsilon_i. \\ &(\partial \varphi_i / \partial r)(s,r) > 0 \quad \text{if } r > 0. \end{aligned} \tag{4.4}$$

For each $1 \le k \le n$, and $0 < \delta_0 < 1$, we define a map

$$\varphi(k) : [0,1] \times E(k)_{\delta_0} \to E(k)_{\delta_0} \quad \text{by} \quad \varphi(s, (r_i)_{i \in A(k)}) := (\varphi_i(s, r_i))_{i \in A(k)}. \tag{4.5}$$

Then, for any fixed point $(r_i)_{i\in A(k)} \in C(k)_{\delta_0}$ and a fixed non-negative number $\delta \le \delta_0$, the curve $\varphi([0,1],(r_i)_{i\in A(k)})$ and the hypersurface $C(k)_\delta$ intersect at one point and, moreover, they are transversal except at the points of the singular locus of $C(k)_0$. Denote this intersection point by

(4.6) $$\langle r,(r_i)_{i\in A(k)}\rangle, \quad \text{where} \quad r := \delta/\delta_0,$$

and call this the *hyperbolic polar coordinates* of the point in $E(k)_{\delta_0}$. Define

(4.7) $$R(k) : [0,1]\times E(k)_{\delta_0} \to E(k)_{\delta_0} \quad \text{by} \quad R(s,\langle r,(r_i)_{i\in A(k)}\rangle) := \langle sr,(r_i)_{i\in A(k)}\rangle.$$

Here we may assume that the above number δ_0 is chosen so small that, for every $1\le k\le n$,

(4.8) $$(r_i)_{i\in A(k)} \in E(k)_{\delta_0} \text{ implies } r_i < \varepsilon_i/2 \text{ for some } i\in A(k).$$

Then, for each $1\le k\le n$,

(4.9) $$\{(r_i)_{i\in A(k)} \in C(k)_{\delta_0} \mid r_j < \varepsilon_j/2\}_{j\in A(k)}$$

forms an open covering of $C(k)_{\delta_0}$. Take a C^∞ partition of unity

(4.10) $$\{\lambda_j\}_{j\in A(k)}$$

on $C(k)_{\delta_0}$ which is subordinate to the covering (4.9), and extend this over $E(k)_{\delta_0}$ by

$$\lambda_j(\langle r,(r_i)_{i\in A(k)}\rangle) := \lambda_j((r_i)_{i\in A(k)}) \quad \text{for all } r\in[0,1].$$

Now we define an action of the monoid S on $X^{\log}$ in the following way. For the C^∞ global equations z_i of D_i $(i\in A-A(0))$ in Proposition 3 (b), let

(4.11) $$z_i(y) =: r_i(y)u_i(y), \quad y\in X,$$

be the decompositions into the absolute values and the arguments. We choose the positive numbers ε_i in (4.2) so small that $\{y\in X \mid r_i(y)\in\varepsilon_i\}$ is contained in the tubular neighborhood U_i in Proposition 3 $(i\in A-A(0))$, and we shrink the polydisc $P=\Delta^n$ so that $X\subset\bigcup_{i\in A-A(0)}U_i$, $r_i(y)\le 1$ $(y,\in X, i\in A-A(0))$ and the radius of each factor Δ is less than or equal δ_0. For $y\in X$, let

(4.12) $$\begin{aligned} &I := \{i\in A-A(0) \mid U_i\ni y\}, \quad x := \pi_I(y), \quad F := \pi_I^{-1}(x),\\ &F^{\log} : \text{the closure of } \tau_X^{-1}(F-F\cap D) \text{ in } X^{\log}\end{aligned}$$

We define an action $S\times F^{\log}\to F^{\log}$ by

(4.13) $$\begin{aligned} \big(r_i((s,v)\cdot\tilde{y})\big)_{i\in A(k)} &:= R(k)\big(s(k),(r_j(\tilde{y}))_{j\in A(k)}\big),\\ u_i((s,v)\cdot\tilde{y}) &:= v(k)^{\lambda_i(\tilde{y})/m(i)}u_i(\tilde{y}) \qquad (i\in A(k))\end{aligned}$$

for $1\le k\le n$, where

$$(s,v) = (s(k),v(k))_{1\le k\le n}\in S = ([0,1]\times\mathbf{C}_1)^n, \quad \lambda_i(\tilde{y}) := \lambda_i((r_j(\tilde{y}))_{j\in A(k)}) \quad (i\in A(k)).$$

Then we can verify the following claim:

Claim (4.14) The monoid action (4.13) is compatible with the restricted morphism $f^{\log} : F^{\log} \longrightarrow P^{\log}$, and these actions on the fibers $F^{\log}$ fit together to give a continuous action on $X^{\log}$.

The S-action on $X^{\log}$ preserves the subspace $Y^{\log}$ by Proposition 3 (iv), and they drop down to induce S-actions on X and on Y. We see that these S-actions are compatible with the natural ones on P and on $P^{\log}$. Let $O \in P$ be the origin. We denote

$$(4.15) \qquad O^{\log} := \tau_P^{-1}(O) \simeq (\mathbf{C}_1)^n, \qquad X^{\log}_{O^{\log}} := (f^{\log})^{-1}(O^{\log}).$$

For $(\mathbf{0}, \mathbf{1}) = ((0, \ldots, 0), (1, \ldots, 1)) \in S$, we define a continuous map

$$(4.16) \qquad \tilde{\pi} : X^{\log} \longrightarrow X^{\log}_{O^{\log}} \quad \text{by} \quad \tilde{\pi}(\tilde{y}) := (\mathbf{0}, \mathbf{1}) \cdot \tilde{y}.$$

By Proposition 3 (iv), $\tilde{\pi}$ is compatible with the inclusion $Y^{\log} \subset X^{\log}$. Let $\tilde{t} \in P^{\log}$ and $\tilde{t}_0 := (\mathbf{0}, \mathbf{1}) \cdot \tilde{t} \in O^{\log}$, and let $X^{\log}_{\tilde{t}}$ and $X^{\log}_{\tilde{t}_0}$ be the fibers of $f^{\log}$ over $\tilde{t}$ and $\tilde{t}_0$, respectively. Then we can verify the following claim:

Claim (4.17) The restricted map $\tilde{\pi} : X^{\log}_{\tilde{t}} \longrightarrow X^{\log}_{\tilde{t}_0}$ is homeomorphic.

From this, we see that the map $\tilde{\pi}$ in (4.16) yields a horizontal projection of the family $f^{\log} : X^{\log} \longrightarrow P^{\log}$, compatible with the inclusion $Y^{\log} \subset X^{\log}$. Thus we get Theorem 1.

The above argument is essentially the same as in the case of the $\dim P = 1$ and the details in this case can be found in [U, §3].

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